ORBIT COUNTING WITH AN ISOMETRIC DIRECTION

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ABSTRACT. Analogues of the prime number theorem and Merten's theorem are well-known for dynamical systems with hyperbolic behaviour. In this paper a 3-adic extension of the circle doubling map is studied. The map has a 3-adic eigendirection in which it behaves like an isometry, and the loss of hyperbolicity leads to weaker asymptotic results on orbit counting than those obtained for hyperbolic maps.

1. INTRODUCTION

As part of the analogy between the behaviour of closed orbits in hyperbolic dynamical systems and prime numbers in [11] both the prime number theorem [9], [10] and Merten's theorem [13] have corresponding results. Roughly speaking, these results mean the following for a hyperbolic transformation $f : X \to X$ with topological entropy $h = h(f)$. A *closed orbit* τ of *length* $|\tau| = n$ is a set of the form

$$\{x, f(x), f^2(x), \dots, f^n(x) = x\}$$

with cardinality n. The analogue of the prime number theorem states that

$$\pi_f(X) = \#\{\tau \colon |\tau| \le X\} \sim \frac{e^{h(X+1)}}{X(e^h - 1)}, \tag{1}$$

and the analogue of Merten's theorem states that

$$\sum_{|\tau| \le X} \frac{1}{e^{h|\tau|}} \sim \log X + C. \tag{2}$$

The results in the papers cited are more precise, and apply to flows as well as hyperbolic maps. We will use the word 'hyperbolicity' rather loosely, and in particular will apply it to group endomorphisms whose

2000 *Mathematics Subject Classification.* 58F20, 11N05.

The second author thanks the EPSRC for their support via a Doctoral Training Award, and the third author thanks Richard Sharp for telling him about [4], and the Max Planck Institute in Bonn where this paper was completed.

natural invertible extension is an expansive automorphism of a finite-dimensional group. If one allows p-adic as well as complex eigenvalues, this corresponds to having no eigenvalues of unit size, and taking the invertible extension does not affect the number of periodic orbits.

Without hyperbolicity, less is known. For ergodic toral automorphisms that are not hyperbolic (the *quasihyperbolic* automorphisms of [5]), Waddington [15] has shown an analogue of (1) and Noorani [8] an analogue of (2). In both cases additional terms appear, but the basic shape remains the same, and the technique of exploiting a meromorphic extension of the dynamical zeta function beyond its radius of convergence still works. In a different direction, Knieper [4] finds asymptotic upper and lower bounds for the function counting closed geodesics on rank-1 manifolds of non-positive curvature of the form

$$A\frac{e^{hX}}{X} \le \#\{\text{closed geodesics of length} \le X\} \le Be^{hX}$$

for constants $A, B > 0$.

Our purpose here is to describe for a specific map how (1) and (2) change when hyperbolicity is lost via the introduction of an isometric extension that gives the local structure of the dynamical system a (non-Archimedean) isometric direction. This work is a special case of more general results in [14], where many isometric directions, a positive-characteristic analogue, and more refined estimates are considered. The essential issues that arise are well illustrated by the map studied here, which avoids many technicalities. In particular, the bad behaviour of the dynamical zeta function appears for this simple example already.

2. Orbits and periodic points

We first recall some basic properties of maps and their periodic points (see [12] for example). Let $T : X \to X$ be a map. The number of points of period n under T is

$$\mathcal{F}_n(T) = \#\{x \in X \colon T^n x = x\},$$

the number of periodic points with least period n under T is

$$\mathcal{L}_n(T) = \#\{x \in X \colon T^n x = x \text{ and } \#\{T^k x\}_{k\in\mathbb{N}} = n\},$$

and the number of closed orbits of length n is

$$\mathcal{O}_n(T) = \mathcal{L}_n(T)/n. \tag{3}$$

Since

$$\mathcal{F}_n(T) = \sum_{d|n} \mathcal{L}_d(T), \tag{4}$$

the sequence of values taken by any one of these quantities determines the others. A consequence of this is a relationship between the number of orbits of a map and the number of orbits of its iterates which will be needed in a special case later.

Lemma 1. *For any* $k \geq 1$, $\mathcal{O}_n(T^k) = \frac{1}{n}\sum_{d|n}\mu(n/d)\sum_{d'|dk} d'\,\mathcal{O}_{d'}(T)$. *In particular,*

$$\mathcal{O}_n(T^2) = \begin{cases} 2\,\mathcal{O}_{2n}(T) + \mathcal{O}_n(T) & \text{if } n \text{ is odd,} \\ 2\,\mathcal{O}_{2n}(T) & \text{if } n \text{ is even.} \end{cases}$$

Proof. Clearly $\mathcal{F}_d(T^k) = \mathcal{F}_{dk}(T)$ for all $d \geq 1$ so, by Möbius inversion of (4),

$$\begin{aligned} \mathcal{L}_n(T^k) &= \sum_{d|n}\mu(n/d)\,\mathcal{F}_d(T^k) = \sum_{d|n}\mu(n/d)\,\mathcal{F}_{kd}(T) \\ &= \sum_{d|n}\mu(n/d)\sum_{d'|dk}\mathcal{L}_{d'}(T), \end{aligned}$$

which gives the result by (3).

The expression for $\mathcal{O}_n(T^2)$ may be checked as follows. First notice that by (4) it is sufficient to check that the given expression gives the right value to $\mathcal{F}_n(T^2)$. If n is odd, then all factors of n are odd so

$$\begin{aligned} \mathcal{F}_n(T^2) &= \sum_{d|n} d\,\mathcal{O}_d(T^2) \\ &= \sum_{d|n} 2d\,\mathcal{O}_{2d}(T) + \sum_{d|n} d\,\mathcal{O}_d(T) \\ &= \sum_{d|2n} d\,\mathcal{O}_d(T) = \mathcal{F}_{2n}(T) \end{aligned}$$

as required. If n is even, then

$$
\begin{aligned}
\mathcal{F}_n(T^2) &= \sum_{d|n} d\,\mathcal{O}_d(T^2) \\
&= \sum_{2|d|n} 2d\,\mathcal{O}_{2d}(T) + \sum_{2\nmid d|n} 2d\,\mathcal{O}_{2d}(T) + \sum_{2\nmid d|n} d\,\mathcal{O}_d(T) \\
&= \sum_{d|n} 2d\,\mathcal{O}_{2d}(T) + \sum_{2\nmid d|n} d\,\mathcal{O}_d(T) \\
&= \sum_{2|d|2n} d\,\mathcal{O}_d(T) + \sum_{2\nmid d|2n} d\,\mathcal{O}_d(T) \\
&= \sum_{d|2n} d\,\mathcal{O}_d(T) = \mathcal{F}_{2n}(T).
\end{aligned}
$$

□

3. A $\mathbb{Z}_3$-extension of the circle-doubling map

Consider the map $\phi : x \mapsto 2x$ on the ring $\mathbb{Z}[\frac{1}{3}]$. Write $X = \widehat{\mathbb{Z}[\frac{1}{3}]}$ for the dual (character) group, and $f = \widehat{\phi}$ for the dual map. The pair (X, f) is the map we will study here, using the following properties from [2] and [16].

- X is a compact metrizable group and f is a continuous endomorphism of X.
- Locally, the action of f is isometric to the map $(s,t) \mapsto (2s, 2t)$ on an open set in $\mathbb{R} \times \mathbb{Q}_3$.
- In this local picture, the real coordinate is stretched by 2 while the action on the 3-adic coordinate is an isometry.
- The number of points of period n under f is $|2^n - 1| \cdot |2^n - 1|_3$.
- The topological entropy of f is $\log 2$.

The map f may also be thought of as an extension of the familiar circle-doubling map by a cocycle taking values in $\mathbb{Z}_3$, the 3-adic integers. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\imath} \mathbb{Z}[\tfrac{1}{3}] \longrightarrow \mathbb{Z}[\tfrac{1}{3}]/\mathbb{Z} \longrightarrow 0 \tag{5}$$

(where $\imath$ is the inclusion map) commutes with $x \mapsto 2x$. The dual of (5) is the short exact sequence

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow X \xrightarrow{\hat{\imath}} \mathbb{T} \longrightarrow 0$$

which commutes with the dual of $x \mapsto 2x$. This sequence expresses $f : X \to X$ as an extension of the circle-doubling map by a cocycle taking values in $\mathbb{Z}_3$. This extension kills certain periodic orbits.

4. Dynamical zeta function

From now on $f : X \to X$ is the map from the previous section. The arguments below are elementary: the most sophisticated number-theoretic fact needed is that the divisors of an odd number are all odd.

First notice that

$$\begin{aligned} |2^n - 1|_3 &= |(3-1)^n - 1|_3 \\ &= |3^n - n3^{n-1} + \cdots + (-1)^{n-1}3n + (-1)^n - 1|_3 \\ &= \begin{cases} \frac{1}{3}|n|_3 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

In particular,

$$|4^n - 1|_3 = |2^{2n} - 1|_3 = \tfrac{1}{3}|2n|_3 = \tfrac{1}{3}|n|_3. \tag{6}$$

Since $|n|_3 = 3^{-\operatorname{ord}_3(n)} \geq 3^{-\log_3(n)} \geq 1/n$, it follows that

$$\tfrac{1}{3n} \leq |2^n - 1|_3 \leq 1 \text{ for all } n \geq 1, \tag{7}$$

so just as for the circle doubling map the logarithmic growth rate of periodic points gives the topological entropy,

$$\frac{1}{n} \log \mathcal{F}_n(f) \longrightarrow \log 2 = h(f).$$

Thus the radius of convergence of the dynamical zeta function

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \mathcal{F}_n(f)$$

is $\exp(-h(f)) = \frac{1}{2}$.

The bound (7) means that the number of periodic points for the circle doubling map is only polynomially larger than the number of periodic points for f. This is enough room to make a real difference: for example, in contrast to the hyperbolic case, $\frac{\mathcal{F}_{n+1}(f)}{\mathcal{F}_n(f)}$ does not converge as $n \to \infty$ (indeed, for S-integer systems in zero characteristic, this convergence essentially characterizes hyperbolicity, cf. [2, Th. 6.3]).

One of the key tools in the hyperbolic case is a meromorphic extension of the zeta function to a larger disc; in our setting this is impossible.

Proposition 2. *The dynamical zeta function of f has natural boundary $|z| = \frac{1}{2}$.*

Proof. Let $\xi(z)=\sum_{n=1}^{\infty}\frac{z^n}{n}|2^n-1|\cdot|2^n-1|_3$ so $\zeta(z)=\exp(\xi(z))$. Now

$$\begin{aligned}\xi(z) &= \sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}(2^{2n+1}-1)+\sum_{n=1}^{\infty}\frac{z^{2n}}{2n}(2^{2n}-1)|2^{2n}-1|_3\\ &= \log\left(\frac{1-z}{1-2z}\right)-\frac{1}{2}\log\left(\frac{1-z^2}{1-4z^2}\right)+\sum_{n=1}^{\infty}\frac{z^{2n}}{2n}(2^{2n}-1)|2^{2n}-1|_3.\end{aligned}$$

Write $\frac{1}{6}\xi_1(z)$ for the last term in this expression, so

$$\begin{aligned}\xi_1(z) &= 3\sum_{n=1}^{\infty}\frac{z^{2n}}{n}(4^n-1)|4^n-1|_3\\ &= \sum_{n=1}^{\infty}\frac{z^{2n}}{n}(4^n-1)|n|_3\end{aligned}$$

by (6). We shall show that $\xi_1(z)$ has infinitely many logarithmic singularities on the circle $|z|=\frac{1}{2}$, each of which corresponds to a zero of $\zeta(z)$.

Write $3^a\|n$ to mean that $3^a\big|n$ but $3^{a+1}\not|\,n$. Notice that $3^a\|n$ if and only if $|n|_3=3^{-a}$. Then ξ_1 may be split up according to the size of $|n|_3$ as

$$\begin{aligned}\xi_1(z) &= \sum_{j=0}^{\infty}\frac{1}{3^j}\sum_{3^j\|n}\frac{z^{2n}}{n}(4^n-1)\\ &= \sum_{j=0}^{\infty}\frac{1}{3^j}\eta_j^{(4)}(z),\end{aligned}$$

where

$$\eta_j^{(a)}(z)=\sum_{3^j\|n}\frac{z^{2n}}{n}(a^n-1).$$

Then

$$\begin{aligned}\eta_0^{(a)}(z) &= \sum_{3^0\|n}\frac{z^{2n}}{n}(a^n-1)\\ &= \sum_{n=1}^{\infty}\frac{z^{2n}}{n}(a^n-1)-\sum_{n=1}^{\infty}\frac{z^{6n}}{3n}(a^{3n}-1)\\ &= \log\left(\frac{1-z^2}{1-az^2}\right)-\frac{1}{3}\log\left(\frac{1-z^6}{1-a^3z^6}\right),\end{aligned}$$

$$\eta_1^{(4)}(z)=\sum_{3^1\|n}\frac{z^{2n}}{n}(4^n-1)=\sum_{3^0\|n}\frac{z^{6n}}{3n}(4^{3n}-1)=\tfrac{1}{3}\eta_0^{(4^3)}(z^3),$$

$$\eta_2^{(4)}(z) = \tfrac{1}{9}\eta_0^{(4^9)}(z^9),$$

and so on. Thus

$$\xi_1(z) = \log\left(\frac{1-z^2}{1-(2z)^2}\right) + 2\sum_{j=1}^{\infty}\frac{1}{9^j}\log\left(\frac{1-(2z)^{2\times 3^j}}{1-z^{2\times 3^j}}\right),$$

so

$$|\zeta(z)| = \left|\frac{1-z}{1-2z}\right| \cdot \left|\frac{1-(2z)^2}{1-z^2}\right|^{1/2} \cdot \left|\frac{1-z^2}{1-(2z)^2}\right|^{1/6} \cdot \prod_{j=1}^{\infty}\left|\frac{1-(2z)^{2\times 3^j}}{1-z^{2\times 3^j}}\right|^{1/3\times 9^j}$$

It follows that the series defining $\zeta(z)$ has a zero at all points of the form $\frac{1}{2}e^{2\pi i j/3^r}$, $r \geq 1$ so $|z| = \frac{1}{2}$ is a natural boundary for $\zeta(z)$. □

A natural boundary appears for most of the S-integer dynamical systems studied in [2], but only *ad hoc* proofs of this exist. Natural boundaries also arise for certain 'random' zeta functions [1] and in zeta functions for higher-rank actions [6]. The appearance of singularities at all g^rth unit roots creating a natural boundary has been exploited by Hecke and Mahler in other contexts [3], [7].

5. Prime orbit theorem

The first result shows the asymptotics for $\pi_f(X)$, and is arrived at by comparison with the well-understood asymptotics for the circle-doubling map $g : \mathbb{T} \to \mathbb{T}$, $x \mapsto 2x \bmod 1$. Here $\mathcal{F}_n(g) = 2^n - 1$, $\mathcal{O}_n(g) = \frac{1}{n}\sum_{d|n}\mu(n/d)(2^d-1)$, and (1) applies to show that

$$\pi_g(X) = \sum_{n\leq X}\frac{1}{n}\sum_{d|n}\mu(n/d)(2^d-1) \sim \frac{2^{X+1}}{X}. \tag{8}$$

Pursuing the analogy between results like (8) and the prime number theorem, the next result is analogous to Tchebycheff's theorem.

Theorem 3.

$$\pi_f(X) \leq \pi_g(X) \textit{ for all } X \geq 1,$$

and

$$\limsup_{X\to\infty}\frac{X\pi_f(X)}{2^{X+1}} \leq 1, \quad \liminf_{X\to\infty}\frac{X\pi_f(X)}{2^{X+1}} \geq \frac{1}{3}.$$

Proof. Let $b_n = 2^n - 1$ and $a_n = b_n|2^n-1|_3$, so

$$\pi_f(X) = \sum_{n\leq X}\mathcal{O}_n(f) = \sum_{n\leq X}\frac{1}{n}\sum_{d|n}\mu(n/d)a_d.$$

We first claim that

$$\mathcal{O}_n(f) \leq \mathcal{O}_n(g) \text{ for all } n \geq 1. \tag{9}$$

Notice that this does not follow *a priori* from the fact that

$$\mathcal{F}_n(f) \le \mathcal{F}_n(g) \text{ for all } n \ge 1;$$

it is easy to construct pairs of maps with one of these inequalities but not the other (see Example 4). However, (9) does hold here because of the exponential growth in a_n.

For odd n, $a_n = b_n$ so $\mathcal{O}_n(f) = \mathcal{O}_n(g)$ (since all factors of n are also odd).

Now assume that n is even and notice that

$$\sum_{d|n, d<n} (2^d - 1) \le \tfrac{2}{3}(2^n - 1) \text{ for all } n \ge 1. \tag{10}$$

It follows that

$$\begin{aligned} \mathcal{L}_n(f) &= \sum_{d|n} \mu(n/d) a_d \\ &\le a_n + \sum_{d|n, d<n} a_d \\ &\le \tfrac{1}{3} b_n + \textstyle\sum_{d|n,d<n} b_d \\ &\le \sum_{d|n} \mu(n/d) b_d = \mathcal{L}_n(g) \end{aligned}$$

by (10). Thus $\mathcal{O}_n(f) \le \mathcal{O}_n(g)$ for all $n \ge 1$, so (9) – and hence the upper bound in Theorem 3 – is proved.

Turning to the lower bound, write

$$\delta(X) = \pi_g(X) - \pi_f(X) \ge 0,$$

and notice that

$$\delta(X) = \sum_{n \le X} (\mathcal{O}_n(g) - \mathcal{O}_n(f)) = \sum_{2|n \le X} (\mathcal{O}_n(g) - \mathcal{O}_n(f)) \le \sum_{2|n \le X} \mathcal{O}_n(g).$$

So we need to estimate the size of $\sum_{2|n\le X} \mathcal{O}_n(g)$. Notice that

$$\mathcal{F}_n(g^2) = 4^n - 1$$

and g^2 (the map $x \mapsto 4x \bmod 1$ on the circle) is hyperbolic, so

$$\sum_{n \le X} \mathcal{O}_n(g^2) \sim \frac{4^{X+1}}{3X}.$$

By Lemma 1,

$$\mathcal{O}_n(g^2) = \begin{cases} 2\,\mathcal{O}_{2n}(g) + \mathcal{O}_n(g) & \text{if } n \text{ is odd,} \\ 2\,\mathcal{O}_{2n}(g) & \text{if } n \text{ is even.} \end{cases}$$

So

$$\begin{aligned} 2\sum_{n\le X}\mathcal{O}_{2n}(g) &= \sum_{2\nmid n\le X}\big(\mathcal{O}_n(g^2)-\mathcal{O}_n(g)\big)+\sum_{2\mid n\le X}\mathcal{O}_n(g^2)\\ &= \sum_{n\le X}\mathcal{O}_n(g^2)-\sum_{2\nmid n\le X}\mathcal{O}_n(g). \end{aligned}$$

Now $\sum_{n\le X}\mathcal{O}_n(g^2)\sim\frac{4^{X+1}}{3X}$, so

$$\sum_{n\le X}\mathcal{O}_{2n}(g)\sim\frac{2}{3}\cdot\frac{4^X}{X}-\frac{1}{2}\sum_{2\nmid n\le X}\mathcal{O}_n(g).$$

On the other hand, $\sum_{n\le X}\mathcal{O}_n(g)\sim\frac{2^{X+1}}{X}$, so the last term is of lower order. It follows that

$$\sum_{n\le X}\mathcal{O}_{2n}(g)\sim\frac{2}{3}\cdot\frac{4^X}{X},$$

so

$$\pi_g(X)-\pi_f(X)\le\sum_{2\mid n\le X}\mathcal{O}_n(g)\sim\frac{2}{3}\cdot\frac{4^{X/2}}{X/2}=\frac{2}{3}\cdot\frac{2^{X+1}}{X}.$$

Thus

$$\limsup_{X\to\infty}\frac{X\pi_f(X)}{2^{X+1}}\le 1,\quad \liminf_{X\to\infty}\frac{X\pi_f(X)}{2^{X+1}}\ge\frac{1}{3}.$$

□

Finding the exact values of the upper and lower limits requires more careful estimates; numerical evidence suggests the limit does not exist, and moreover that the sequence $\left(\frac{X\pi_f(X)}{2^{X+1}}\right)_{X\ge1}$ has more than two limit points.

In the proof of Theorem 3 we mentioned that for a pair of maps f,g, $\mathcal{F}_n(f)\le\mathcal{F}_n(g)$ for all $n\ge1$ does not imply that $\mathcal{O}_n(f)\le\mathcal{O}_n(g)$. Of course

$$\mathcal{O}_n(f)\le\mathcal{O}_n(g)\text{ for all }n\ge1\implies\mathcal{F}_n(f)\le\mathcal{F}_n(g)\text{ for all }n\ge1$$

by (3) and (4).

Example 4. Given any sequence $(o_n)_{n\ge1}$ of non-negative integers, there is a map with o_n closed orbits of length n (indeed, there is a C^∞ map of the 2-torus with this property by [17]). Let f be chosen with $\mathcal{O}_1(f)=1$, $\mathcal{O}_2(f)=3$ and $\mathcal{O}_n(f)=0$ for $n\ge3$. Similarly, let g be chosen with $\mathcal{O}_1(g)=6$, $\mathcal{O}_2(g)=1$ and $\mathcal{O}_n(g)=0$ for $n\ge3$. Then $\mathcal{O}_n(f)$ is not bounded above by $\mathcal{O}_n(g)$, but

$$\mathcal{F}_n(f)=4+3\cdot(-1)^n<\mathcal{F}_n(g)=7+(-1)^n\text{ for all }n\ge1.$$

6. Merten's theorem

For the circle doubling map g, the dynamical analogue of Merten's theorem says (in a weak formulation)

$$\sum_{n\le X}\frac{\mathcal{O}_n(g)}{2^n}=\log X+O(1). \tag{11}$$

In our setting, this asymptotic is lost in much the same way as Theorem 3 loses the single asymptotic (1)

Theorem 5.

$$\tfrac{1}{2}\log X+O(1)\le\textstyle\sum_{n\le X}\frac{\mathcal{O}_n(f)}{2^n}\le\log X+O(1).$$

Proof. From the proof of Theorem 3, $\mathcal{O}_n(f)\le\mathcal{O}_n(g)$ for all $n\ge 1$, so the upper asymptotic is an immediate consequence of (11).

To get a lower bound, notice that

$$\sum_{n\le X}\frac{\mathcal{O}_n(f)}{2^n}=\sum_{2\nmid n\le X}\frac{\mathcal{O}_n(f)}{2^n}+\sum_{2|n\le X}\frac{\mathcal{O}_n(f)}{2^n}\ge\sum_{2\nmid n\le X}\frac{\mathcal{O}_n(g)}{2^n}=A(X).$$

This may be estimated using Lemma 1 again. First,

$$A(X)+\sum_{2|n\le X}\frac{\mathcal{O}_n(g)}{2^n}=\sum_{n\le X}\frac{\mathcal{O}_n(g)}{2^n}=\log X+O(1).$$

By Lemma 1,

$$\begin{aligned}
2\sum_{2|n\le X}\frac{\mathcal{O}_n(g)}{2^n} &= 2\sum_{m\le X/2}\frac{\mathcal{O}_{2m}(g)}{2^{2m}}\\
&= \sum_{2\nmid m\le X/2}\frac{\mathcal{O}_m(g^2)-\mathcal{O}_m(g)}{2^{2m}}+\sum_{2|m\le X/2}\frac{\mathcal{O}_m(g^2)}{2^{2m}}\\
&= \sum_{m\le X/2}\frac{\mathcal{O}_m(g^2)}{2^{2m}}-\sum_{2\nmid m\le X/2}\frac{\mathcal{O}_m(g)}{2^{2m}}\\
&\ge \log\frac{X}{2}+O(1)=\log X+O(1)
\end{aligned}$$

since $\mathcal{O}_n(g)\le 2^n$ implies that

$$\sum_{2\nmid m\le X/2}\frac{\mathcal{O}_m(g)}{2^{2m}}\le\sum_{m\le X/2}\frac{\mathcal{O}_m(g)}{2^{2m}}<\infty.$$

$\square$

Just as in Theorem 3, numerical evidence suggests that the lower asymptotic is genuinely lower than $\log X$.

References

1. J. Buzzi, *Some remarks on random zeta functions*, Ergodic Theory Dynam. Systems **22** (2002), no. 4, 1031–1040. MR **2003g:**37034
2. V. Chothi, G. Everest, and T. Ward, *S-integer dynamical systems: periodic points*, J. Reine Angew. Math. **489** (1997), 99–132. MR **99b:**11089
3. E. Hecke, *Über analytische Funktionen und die Verteilung von Zahlen mod. eins*, Abhandlungen aus dem Mathemischen Sminar der Universitat Hamburg **1** (1922), 54–76.
4. G. Knieper, *On the asymptotic geometry of nonpositively curved manifolds*, Geom. Funct. Anal. **7** (1997), no. 4, 755–782. MR **98h:**53055
5. D. Lind, *Dynamical properties of quasihyperbolic toral automorphisms*, Ergodic Theory Dynam. Systems **2** (1982), no. 1, 49–68. MR **84g:**28017
6. D. A. Lind, *A zeta function for $\mathbb{Z}^d$-actions*, Ergodic theory of $\mathbb{Z}^d$ actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, pp. 433–450. MR **97e:**58185
7. K. Mahler, *On two analytic functions*, Acta Arith. **49** (1987), no. 1, 15–20. MR **89f:**30006
8. M. Noorani, *Mertens theorem and closed orbits of ergodic toral automorphisms*, Bull. Malaysian Math. Soc. (2) **22** (1999), no. 2, 127–133. MR **2001k:**37036
9. W. Parry, *An analogue of the prime number theorem for closed orbits of shifts of finite type and their suspensions*, Israel J. Math. **45** (1983), no. 1, 41–52. MR **85c:**58089
10. W. Parry and M. Pollicott, *An analogue of the prime number theorem for closed orbits of Axiom A flows*, Ann. of Math. (2) **118** (1983), no. 3, 573–591. MR **85i:**58105
11. ———, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque (1990), no. 187-188, 268. MR **92f:**58141
12. Y. Puri and T. Ward, *Arithmetic and growth of periodic orbits*, J. Integer Seq. **4** (2001), no. 2, Article 01.2.1, 18 pp. MR **2002i:**11026
13. R. Sharp, *An analogue of Mertens' theorem for closed orbits of Axiom A flows*, Bol. Soc. Brasil. Mat. (N.S.) **21** (1991), no. 2, 205–229. MR **93a:**58142
14. V. Stangoe, *Orbit counting far from hyperbolicity*, Ph.D. thesis, University of East Anglia, 2004.
15. S. Waddington, *The prime orbit theorem for quasihyperbolic toral automorphisms*, Monatsh. Math. **112** (1991), no. 3, 235–248. MR **92k:**58219
16. T. Ward, *Almost all S-integer dynamical systems have many periodic points*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 471–486. MR **99k:**58152
17. A. Windsor, *Smoothness is not an obstruction to exact realizability*, Preprint, 2003.

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